

Nichols algebras over solvable groups

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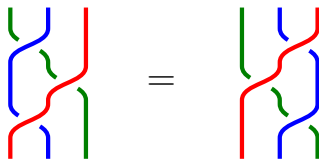
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Braided vector spaces

Let (V, c) be a **braided vector space**, that is a K -vector space V with a bijective linear map $c \in \mathbf{GL}(V \otimes V)$ that satisfies the **braid equation**:

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).$$



Example 1:

Let V be a vector space with basis $\{x_1, x_2, \dots, x_\theta\}$. Then

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i,$$

where $q_{ij} \in K^\times$, is a **braiding** (of **diagonal type**).

Example 2:

Let G be a group and X be a union of conjugacy classes of G . Let V be a K -vector space with basis X . Then

$$c(x \otimes y) = q_{xy}xyx^{-1} \otimes x,$$

where $q_{xy} \in K^\times$ is a certain collection of scalars, is a **braiding** (of **group type**).

Braid groups

The **braid group** \mathbb{B}_n has generators $\sigma_1, \dots, \sigma_{n-1}$ and relations

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \leq i \leq n-2, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i & i - j \geq 2.\end{aligned}$$

If (V, c) is a braided vector space, then

$$\rho: \mathbb{B}_n \rightarrow \mathbf{GL}(V^{\otimes n}), \quad \sigma_i \mapsto c_i,$$

is a group homomorphism, where

$$c_k = \mathrm{id}^{\otimes(k-1)} \otimes c \otimes \mathrm{id}^{\otimes(n-k-1)}.$$

Braid groups

For example, let $n = 4$. If we represent the braiding c by the diagram



then the maps c_1 and c_2 are given by



respectively.

A braided vector space (V, c) gives a special type of algebra called the **Nichols algebra** $\mathcal{B}(V, c)$.

Nichols algebras

The **Nichols algebra** of (V, c) is constructed as a quotient of the tensor algebra of V :

$$\mathcal{B}(V, c) = K \oplus V \oplus \bigoplus_{n \geq 2} V^{\otimes n} / \ker \mathfrak{S}_n,$$

where \mathfrak{S}_n is the **quantum symmetrizer**. For example:

$$\mathfrak{S}_2 = \text{id} + c,$$

$$\mathfrak{S}_3 = \text{id} + c_1 + c_2 + c_1 c_2 + c_2 c_1 + c_1 c_2 c_1,$$

$$\vdots$$

$$\mathfrak{S}_{n+1} = (\text{id} \otimes \mathfrak{S}_n)(\text{id} + c_1 + c_1 c_2 + \cdots + c_1 c_2 \cdots c_n).$$

Some well-known examples of Nichols algebras:

- ▶ (V, flip) gives the symmetric algebra.
- ▶ $(V, -\text{flip})$ gives the exterior algebra.

Nichols algebras (also known as Fock spaces) were rediscovered several times: Nichols, Woronowicz, Lusztig, Majid...

Problem

Classify finite-dimensional Nichols algebras.

For applications, the interesting **Nichols algebras** all come from **groups**. Which braided vector spaces should we consider?

Yetter-Drinfeld modules (over groups)

Let G be a group. A **Yetter-Drinfeld module** V over G is a G -graded KG -module

$$V = \bigoplus_{g \in G} V_g$$

such that

$$g \cdot V_h \subseteq V_{ghg^{-1}}$$

for all $g, h \in G$.

Fact:

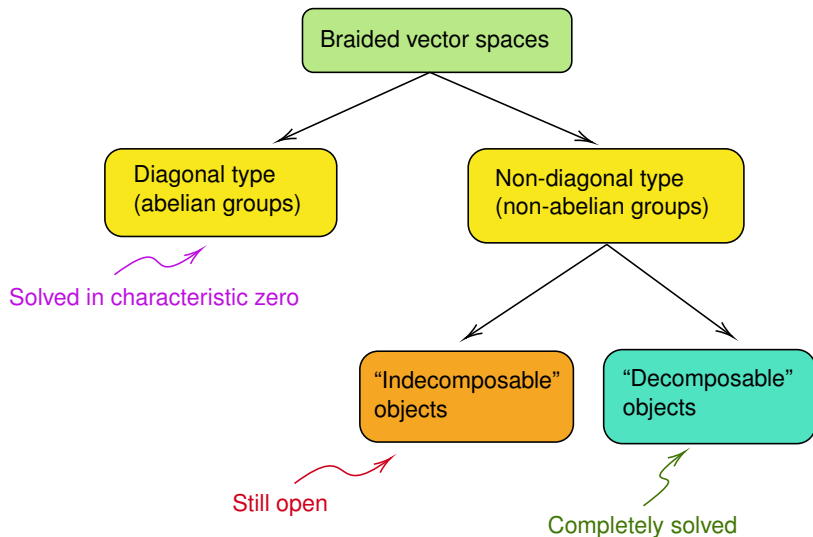
A Yetter-Drinfeld module V is a **braided vector space**:

$$c(v \otimes w) = g \cdot w \otimes v,$$

where $v \in V_g$ and $w \in V$.

Moreover, the category ${}^{KG}_{KG}\mathcal{YD}$ of Yetter-Drinfeld modules over KG is a **braided tensor category**.

The state of the art



Braided vector spaces of diagonal type

A braided vector space V is of **diagonal type** if there exists a basis $\{v_1, \dots, v_\theta\}$ of V such that

$$c(v_i \otimes v_j) = q_{ij} v_j \otimes v_i, \quad q_{ij} \in K^\times.$$

Nichols algebras of braided vector spaces of **diagonal type** have many interesting properties and applications.

Complex finite-dimensional Nichols algebra of diagonal type:

- ▶ Classified by Heckenberger.
- ▶ Generators and relations: Angiono.
- ▶ Applications to **Hopf algebras**: Andruskiewitsch and Schneider's classification.
- ▶ Applications to **physics**: Semikhatov, Lentner.

What about **non-diagonal** Nichols algebras?

This is important to study combinatorial Schubert calculus and pointed Hopf algebras with non-abelian coradical.

Let (V, c) be a braided vector space not of diagonal type. There are two cases to consider:

1. V is “*indecomposable*”.
2. $V = V_1 \oplus \cdots \oplus V_\theta$ is “*decomposable*”, where $\theta \geq 2$.

The classification problem is solved for “decomposable” Yetter-Drinfeld modules.

What does it mean “decomposable”?

We first start with the case of **two irreducible summands**.

Remark (Graña)

If $c_{W,V}c_{V,W} = \text{id}_{V \otimes W}$ then

$$\mathcal{B}(V \oplus W) \simeq \mathcal{B}(V) \otimes \mathcal{B}(W)$$

as graded vector spaces.

The remark implies that in order to be in the “**decomposable**” **case** one needs to assume that

$$c_{W,V}c_{V,W} \neq \text{id}_{V \otimes W}.$$

Technical definition:

The **support** of a Yetter-Drinfeld module

$$V = \bigoplus_{g \in G} V_g$$

is the set

$$\text{supp} V = \{g \in G : V_g \neq 0\}.$$

Fact:

$\text{supp}(V)$ is a union of conjugacy classes of G .

Theorem (with Heckenberger)

Let G be a non-abelian group, and V and W be two absolutely irreducible Yetter-Drinfeld modules over KG . Assume that

- ▶ G is generated by the support of $V \oplus W$,
- ▶ $c_{W,V}c_{V,W} \neq \text{id}_{V \otimes W}$, and
- ▶ $\dim \mathcal{B}(V \oplus W) < \infty$.

Then G is either an epimorphic image of a certain central extension T of the group $\mathbf{SL}_2(3)$, or an epimorphic image of a certain central extension of the dihedral group of order $2n$ for $n \in \{2, 3, 4\}$.

Non-diagonal type: the “decomposable” case

The theorem has **deep consequences**. One obtains:

- ▶ The **structure** of the braided vector spaces V and W .
- ▶ The **dimension** of $\mathcal{B}(V \oplus W)$.

Theorem (with Heckenberger)

Let G be a non-abelian group, and V and W be two irreducible Yetter-Drinfeld modules over $\mathbb{C}G$. Assume that

- ▶ G is generated by the support of $V \oplus W$,
- ▶ $c_{W,V}c_{V,W} \neq \text{id}_{V \otimes W}$, and
- ▶ $\dim \mathcal{B}(V \oplus W) < \infty$.

Then $\mathcal{B}(V \oplus W)$ is one of following Nichols algebras:

$\dim(V \oplus W)$	$\dim \mathcal{B}(V \oplus W)$
4	64
4 or 5	10368
5	2304
5	80621568
6	262144

An example: the group T

Let us show one of the examples we found (over the complex numbers).

The group T can be presented by generators z, x_1, x_2, x_3, x_4 and relations

$$zx_i = x_iz, \quad i \in \{1, 2, 3, 4\},$$

and

$$x_1x_2 = x_4x_1 = x_2x_4,$$

$$x_1x_3 = x_2x_1 = x_3x_2,$$

$$x_2x_3 = x_4x_2 = x_3x_4,$$

$$x_1x_4 = x_3x_1 = x_4x_3.$$

An example: the module V

Let G be a non-abelian epimorphic image of the group T . We show the structure of the modules V and W .

How does V look like? Let ρ be a character of the centralizer $G^z = G$ and $v \in V_z \setminus \{0\}$. Then $\{v\}$ is basis of V and **the action of G on V** is given by

$$zv = \rho(z)v, \quad x_i v = \rho(x_1)v \quad \text{for all } i \in \{1, 2, 3, 4\}.$$

An example: the module W

How does W look like? Let σ be a character of $G^{x_1} = \langle x_1, x_2x_3, z \rangle$ with $\sigma(x_1) = -1$ and $\sigma(x_2x_3) = 1$. Let $w_1 \in W_{x_1}$ be such that $w_1 \neq 0$. Then the vectors

$$w_1, w_2 = -x_4w_1, w_3 = -x_2w_1, w_4 = -x_3w_1$$

form a basis of W . The degrees of these vectors are x_1, x_2, x_3 and x_4 , respectively. **The action of G on W** is given by the following table:

W	w_1	w_2	w_3	w_4
x_1	$-w_1$	$-w_4$	$-w_2$	$-w_3$
x_2	$-w_3$	$-w_2$	$-w_4$	$-w_1$
x_3	$-w_4$	$-w_1$	$-w_3$	$-w_2$
x_4	$-w_2$	$-w_3$	$-w_1$	$-w_4$
z	$\sigma(z)w_1$	$\sigma(z)w_2$	$\sigma(z)w_3$	$\sigma(z)w_4$

An example: the dimension

Assume further that

$$(\rho(x_1)\sigma(z))^2 - \rho(x_1)\sigma(z) + 1 = 0, \quad \rho(x_1z)\sigma(z) = 1.$$

Then

$$\dim \mathcal{B}(V \oplus W) = 6^3 72^3 = 80621568.$$

The theorem holds for arbitrary characteristic!

The Nichols algebras of the classification

$\dim(V \oplus W)$	$\dim \mathcal{B}(V \oplus W)$	characteristic
4	64	
4	1296	3
4 or 5	10368	$\neq 2, 3$
4 or 5	5184	2
4 or 5	1152	3
4 or 5	2239488	2
5	2304	
5	80621568	$\neq 2$
5	1259712	2
6	262144	$\neq 2$
6	65536	2

We now study the case of **at least three irreducible** summands.

To be in the “**decomposable**” case, we need to assume that

$$M = (M_1, \dots, M_\theta)$$

is **connected**, i.e. $M_1 \oplus \dots \oplus M_\theta$ admits no decomposition

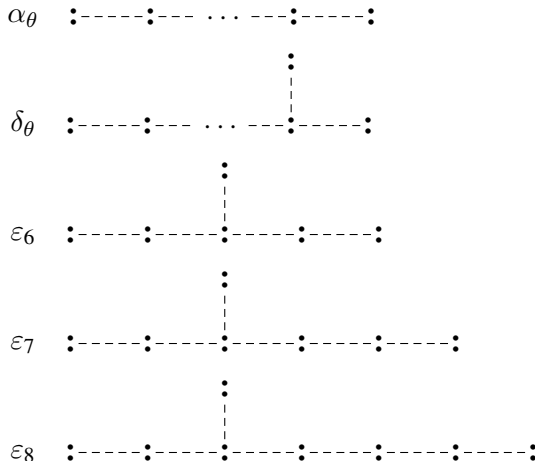
$$M_1 \oplus \dots \oplus M_\theta = M' \oplus M''$$

as Yetter-Drinfeld modules over G with $M' \neq 0$, $M'' \neq 0$ and $c_{M'', M'} c_{M', M''} = \text{id}$.

We need to introduce the following terminology.

Skeletons (of finite type). A skeleton (of finite-type) is a decorated **Dynkin diagram (of finite-type)** that encodes the structure of the Yetter-Drinfeld module.

The following are the **simply-laced skeleton of finite-type** (i.e. Dynkin types **ADE**):



The other **skeletons of finite type** are:

$$\beta_\theta \quad \cdot \text{-----} \cdot \text{---} \dots \text{---} \cdot \text{-----} \cdot \text{==} \rightrightarrows \cdot \quad \text{char} K = 3$$

$$\beta'_3 \quad \begin{array}{c} p \quad p^{-1} \quad p \\ \cdot \text{-----} \cdot \end{array} \text{==} \rightrightarrows \cdot \quad (3)_{-p} = 0$$

$$\beta''_3 \quad \begin{array}{c} p \quad p^{-1} \\ \cdot \text{==} \rightrightarrows \cdot \end{array} \text{==} \rightrightarrows \cdot \quad (3)_{-p} = 0$$

$$\gamma_\theta \quad \cdot \text{-----} \cdot \text{---} \dots \text{---} \cdot \text{-----} \cdot \text{==} \xleftarrow{-1} \cdot \quad \text{char} K \neq 2$$

$$\varphi_4 \quad \begin{array}{c} -1 \quad -1 \\ \cdot \text{-----} \cdot \end{array} \text{==} \rightrightarrows \cdot \text{-----} \cdot \quad \text{char} K \neq 2$$

Here $(n)_t = 1 + t + \dots + t^{n-1}$.

Theorem (with Heckenberger)

Let $\theta \geq 3$, G be a non-abelian group and

$$M = (M_1, \dots, M_\theta)$$

be a **connected** tuple of absolutely irreducible Yetter-Drinfeld modules over KG . Then $\dim \mathcal{B}(M_1 \oplus \dots \oplus M_\theta) < \infty$ if and only if M has a skeleton of finite-type.

The theorem gives the dimensions of

$$\mathcal{B}(M) = \mathcal{B}(M_1 \oplus \cdots \oplus M_\theta)$$

and the structure of the M_i can be obtained from the skeletons of finite type.

Example:

In the case where M has a simply-laced skeleton of finite type (Dynkin type **ADE**), the dimensions of the **Nichols algebras** in the classification are

$\dim \mathcal{B}(M)$	$4^{\theta(\theta+1)/2}$	$4^{\theta(\theta-1)}$	4^{36}	4^{63}	4^{120}
skeleton	α_θ	δ_θ	ε_6	ε_7	ε_8

Let us review some results in the “**indecomposable**” case.

Theorem (with Andruskiewitsch, Fantino and Graña)

Let $n \geq 5$ and $G = \mathbb{A}_n$. If $0 \neq V \in \mathbb{C}_G^G \mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.

A similar result is valid for **sporadic simple groups**.

Theorem (with Andruskiewitsch, Fantino and Graña)

Let G be a finite sporadic simple group. If $G \notin \{Fi_{22}, B, M\}$ and $0 \neq V \in \mathbb{C}_G^G \mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.

Question

Let G be the Fischer group Fi_{22} , the Baby Monster B or the Monster M , and let $0 \neq V \in {}^G_G\mathcal{YD}$. Is $\dim \mathcal{B}(V) = \infty$?

Several results concerning Nichols algebras over finite [simple groups of Lie type](#) were found by Andruskiewitsch, Carnovale and García.

Conjecture

Let G be a finite simple group. If $0 \neq V \in {}_{\mathbb{C}G}^{\mathbb{C}G}\mathcal{YD}$, then $\dim \mathcal{B}(V) = \infty$.

An open problem: Fomin–Kirillov algebras

Question

Let $n \geq 6$ and $0 \neq V \in \frac{\mathbb{CS}_n}{\mathbb{CS}_n} \mathcal{YD}$. Is $\dim \mathcal{B}(V) = \infty$?

Remarks:

- ▶ Some partial results are known.
- ▶ The problem is related to **Fomin–Kirillov algebras** and **Schubert calculus**.

An open problem: Fomin–Kirillov algebras

To tackle this problem one studies Nichols algebras associated with [conjugacy classes](#) of symmetric groups.

An open problem: Fomin–Kirillov algebras

For $n \geq 3$, let X_n be the conjugacy class of (12) in the symmetric group \mathbb{S}_n . Let V_n be the complex vector space with basis

$$\{v_g : g \in X_n\}$$

and

$$c(v_g \otimes v_h) = -v_{ghg^{-1}} \otimes v_g.$$

Question

When is $\dim \mathcal{B}(V_n) = \infty$?

An open problem: Fomin–Kirillov algebras

Fact:

$\mathcal{B}(V_n)$ is finite-dimensional for $n \in \{3, 4, 5\}$.

n	$\dim V_n$	$\dim \mathcal{B}(V_n)$
3	3	12
4	6	576
5	10	8294400

Conjectures

- ▶ $\dim \mathcal{B}(V_n) = \infty$ for $n \geq 6$.
- ▶ $\mathcal{B}(V_n)$ is quadratic.

So far only few examples of finite-dimensional Nichols algebras over “indecomposable” braided vector spaces of group type are known!

“indecomposable” case: known examples

$\dim V$	$\dim \mathcal{B}(V)$	characteristic
3	12	
3	432	2
4	36	2
4	72	$\neq 2$
4	5184	
6	576	
6	576	
6	576	
5	1280	
5	1280	
7	326592	
7	326592	
10	8294400	
10	8294400	

Question

Are there other finite-dimensional Nichols algebras?

Theorem (with Heckenberger and Meir)

Let G be a non-abelian group and $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$ be an irreducible of **prime** dimension. Assume that $\text{supp } V$ generates G . Then $\dim \mathcal{B}(V) < \infty$ if and only if $\mathcal{B}(V)$ is one of the following Nichols algebras:

$\dim V$	$\dim \mathcal{B}(V)$
3	12
5	1280
5	1280
7	326592
7	326592

The tools used to prove the previous theorem can be pushed forward to obtain far more general theorems on the structure of finite-dimensional Nichols algebras.

Theorem (with Andruskiewitsch and Heckenberger)

Let G be a non-cyclic **solvable** group and $V \in {}^{\mathbb{C}G}_{\mathbb{C}G}\mathcal{YD}$. Assume that $\text{supp } V$ generates G . If $\dim \mathcal{B}(V) < \infty$, then V is irreducible and $\mathcal{B}(V)$ is one of the following “known” Nichols algebras:

$\dim V$	$\dim \mathcal{B}(V)$	
3	12	
4	72	
4	5184	
5	1280	two algebras
6	576	three algebras
7	326592	two algebras

By combining the previous theorem with the celebrated **lifting method** of Andruskiewitsch and Schneider, one obtains a **Feit–Thompson-like theorem** for pointed Hopf algebras.